Tiling systems and homology of lattices in tree products

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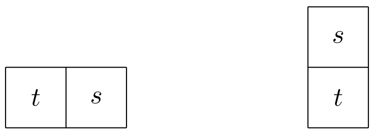
Abstract. Let Γ be a torsion free cocompact lattice in $\mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$, where $\mathcal{T}_1$, $\mathcal{T}_2$ are trees whose vertices all have degree at least three. The group $H_2(\Gamma, \mathbb{Z})$ is determined explicitly in terms of an associated 2-dimensional tiling system. It follows that under appropriate conditions the crossed product C^*-algebra $\mathcal{A}$ associated with the action of Γ on the boundary of $\mathcal{T}_1 \times \mathcal{T}_2$ satisfies $\mathrm{rank}\, K_0(\mathcal{A}) = 2 \cdot \mathrm{rank}\, H_2(\Gamma, \mathbb{Z})$.

1. Introduction

This article is motivated by the problem of calculating the K-theory of certain crossed product C^*-algebras $\mathcal{A}(\Gamma, \partial\Delta)$, where Γ is a higher rank lattice acting on an affine building Δ with boundary $\partial\Delta$. Here we examine the case where Δ is a product of trees. We determine the K-theory rationally, thereby proving some conjectures in [KR].

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be locally finite trees whose vertices all have degree at least three. Consider the direct product $\Delta = \mathcal{T}_1 \times \mathcal{T}_2$ as a two dimensional cell complex. Let Γ be a discrete subgroup of $\mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$ which acts freely and cocompactly on Δ. Associated with the action (Γ, Δ) is a tiling system whose set of tiles is the set $\mathfrak{R}$ of "directed" 2-cells of $\Gamma\backslash\Delta$. There are vertical and horizontal adjacency rules tHs and tVs between tiles $t, s \in \mathfrak{R}$ illustrated below. Precise definitions will be given in Section 2.



There are homomorphisms $T_1, T_2 : \mathbb{Z}\mathfrak{R} \to \mathbb{Z}\mathfrak{R}$ defined by

$$T_1 t = \sum_{tHs} s, \qquad T_2 t = \sum_{tVs} s\,.$$

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Consider the homomorphism $\mathbb{Z}\mathfrak{R} \to \mathbb{Z}\mathfrak{R} \oplus \mathbb{Z}\mathfrak{R}$ given by

$$\begin{pmatrix} T_1 - I \\ T_2 - I \end{pmatrix} : t \mapsto (T_1 t - t) \oplus (T_2 t - t).$$

The main result of this article it the following Theorem, which is formulated more precisely in Theorem 4.1.

Theorem 1.1. *There is an isomorphism*

$$H_2(\Gamma, \mathbb{Z}) \cong \ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix} \right). \tag{1}$$

The proof of (1) is elementary, but care is needed because the right hand side is defined in terms of "directed" 2-cells rather than geometric 2-cells. A square complex X is VH-T if every vertex link is a complete bipartite graph and if there is a partition of the set of edges into vertical and horizontal, which agrees with the bipartition of the graph on every link [BM]. The universal covering space Δ of a VH-T complex X is a product of trees $\mathcal{T}_1 \times \mathcal{T}_2$ and the fundamental group Γ of X is a subgroup of $\mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$ which acts freely and cocompactly on $\mathcal{T}_1 \times \mathcal{T}_2$. Conversely, every finite VH-T complex arises in this way from a free cocompact action of a group Γ on a product of trees. Recall that a discrete group which acts freely on a CAT(0) space is necessarily torsion free.

The group Γ acts on the (maximal) boundary $\partial\Delta$ of Δ, which is the set of chambers of the spherical building at infinity, endowed with an appropriate topology [KR]. This boundary may be identified with a direct product of Gromov boundaries $\partial\mathcal{T}_1 \times \partial\mathcal{T}_2$. The boundary action $(\Gamma, \partial\Delta)$ gives rise to a crossed product C^*-algebra $\mathcal{A}(\Gamma, \partial\Delta) = C_{\mathbb{C}}(\partial\Delta) \rtimes \Gamma$ as described in [KR].

If p is prime then $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on its Bruhat-Tits tree $\mathcal{T}_{p+1}$, which is a homogeneous tree of degree $p+1$. If p, ℓ are prime then the group $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ acts on the $\Delta = \mathcal{T}_{p+1} \times \mathcal{T}_{\ell+1}$. Let Γ be a torsion free irreducible lattice in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$. Then $\mathcal{A}(\Gamma, \partial\Delta)$ is a higher rank Cuntz-Krieger algebra and fits into the general theory developed in [RS1, RS2]. In particular, it is classified up to isomorphism by its K-theory. It is a consequence of Theorem 1.1 (see Section 5) that

$$\operatorname{rank} K_0(\mathcal{A}(\Gamma, \partial\Delta)) = 2 \cdot \operatorname{rank} H_2(\Gamma, \mathbb{Z}). \tag{2}$$

This proves a conjecture in [KR]. The normal subgroup theorem [Mar, IV, Theorem (4.9)] implies that $H_1(\Gamma, \mathbb{Z})$ is a finite group. Equation (2) can therefore be expressed as

$$\chi(\Gamma) = 1 + \frac{1}{2} \operatorname{rank} K_0(\mathcal{A}(\Gamma, \partial\Delta)).$$

One easily calculates that $\chi(\Gamma) = \frac{(p-1)(\ell-1)}{4}|X^0|$, where $|X^0|$ is the number of vertices of X. Therefore the rank of $K_0(\mathcal{A}(\Gamma, \partial\Delta))$ can be expressed explicitly in terms of p, ℓ and $|X^0|$. Examples are constructed in [M3, Section 3], where $p, \ell \equiv 1 \pmod 4$ are two distinct primes.

2. Products of trees and their automorphisms.

If $\mathcal{T}$ is a tree, there is a type map τ defined on the vertex set of $\mathcal{T}$, taking values in $\mathbb{Z}/2\mathbb{Z}$. Two vertices have the same type if and only if the distance between them is even. Any automorphism g of $\mathcal{T}$ preserves distances between vertices, and so there exists $i \in \mathbb{Z}/2\mathbb{Z}$ (depending on g) such that $\tau(gv) = \tau(v) + i$, for every vertex v.

Suppose that Δ is the 2-dimensional cell complex associated with a product $\mathcal{T}_1 \times \mathcal{T}_2$ of trees. Let Δ^k denote the set of k-cells in Δ for $k = 0, 1, 2$. The 0-cells are vertices and the 2-cells are geometric squares. Denote by $u = (u_1, u_2)$ a generic vertex of Δ. There is a type map τ on Δ^0 defined by

$$\tau(u) = (\tau(u_1), \tau(u_2)) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Any 2-cell $\delta \in \Delta^2$ has one vertex of each type. For every $g \in \mathrm{Aut}\mathcal{T}_1 \times \mathrm{Aut}\mathcal{T}_2$ there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex u,

$$\tau(gu) = (\tau(u_1) + k, \tau(u_2) + l). \tag{3}$$

Let $\Gamma < \mathrm{Aut}\mathcal{T}_1 \times \mathrm{Aut}\mathcal{T}_2$ be a torsion free discrete group acting cocompactly on Δ. Then $X = \Gamma\backslash\Delta$ is a finite cell complex with universal covering Δ. Let X^k denote the set of k-cells of X for $k = 0, 1, 2$.

The first step is to formalize the notion of a directed square in X. We modify the terminology of [BM, Section 1], in order to fit with [RS1, RS2, KR]. Let σ be a model typed square with vertices $\mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}$, as illustrated in Figure 2. Assume that the vertex $\mathbf{ij}$ of σ has type

$$\tau(\mathbf{ij}) = (i, j) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

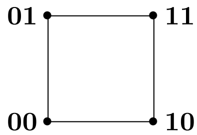


FIGURE 2. The model square σ.

The vertical and horizontal reflections v, h of σ are the involutions satisfying $v(\mathbf{00}) = \mathbf{01}, v(\mathbf{10}) = \mathbf{11}, h(\mathbf{00}) = \mathbf{10}, h(\mathbf{01}) = \mathbf{11}$. An isometry $r : \sigma \to \Delta$ is said to be *type rotating* if there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex $\mathbf{ij}$ of σ

$$\tau(r(\mathbf{ij})) = (i + k, j + l).$$

Let R denote the set of type rotating isometries $r : \sigma \to \Delta$. If $g \in \mathrm{Aut}\mathcal{T}_1 \times \mathrm{Aut}\mathcal{T}_2$ and $r \in R$ then it follows from (3) that $g \circ r \in R$. If $\delta^2 \in \Delta^2$ then for each $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there is a unique $r \in R$ such that $r(\sigma) = \delta^2$ and $r(\mathbf{00})$ has type (k, l). Therefore each geometric square $\delta^2 \in \Delta^2$ is the image of each of the four elements of $\{r \in R \ ; \ r(\sigma) = \delta^2\}$ under the map $r \mapsto r(\sigma)$. The next lemma records this observation.

Lemma 2.1. *The map $r \mapsto r(\sigma)$ from R to Δ^2 is 4-to-1.*

Let $\mathfrak{R} = \Gamma\backslash R$ and call $\mathfrak{R}$ the set of *directed squares* of $X = \Gamma\backslash\Delta$. There is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "r \mapsto r(\sigma)"] \arrow[d] & \Delta^2 \arrow[d] \\ \mathfrak{R} \arrow[r, "\eta"] & X^2 \end{tikzcd}$$

where the vertical arrows represent quotient maps and η is defined by $\eta(\Gamma r) = \Gamma \cdot r(\sigma)$. The next result makes precise the fact that each geometric square in X^2 corresponds to exactly four directed squares.

Lemma 2.2. *The map* $\eta : \mathfrak{R} \to X^2$ *is surjective and* 4*-to-*1*.*

Proof. Fix $\delta^2 \in R$. By Lemma 2.1, the set

$$\{r \in R \ ; \ r(\sigma) = \delta^2\} = \{r_1, r_2, r_3, r_4\}$$

contains precisely 4 elements. Since Γ acts freely on Δ, the set

$$\{\Gamma r_1, \Gamma r_2, \Gamma r_3, \Gamma r_4\} \subset \mathfrak{R}$$

also contains precisely four elements, each of which maps to $\Gamma\delta^2$ under η. Now suppose that $\eta(\Gamma r) = \Gamma\delta^2$ for some $r \in R$. Then $\gamma r(\sigma) = \delta^2$ for some $\gamma \in \Gamma$. Thus $\gamma r \in \{r_1, r_2, r_3, r_4\}$ and $\Gamma r \in \{\Gamma r_1, \Gamma r_2, \Gamma r_3, \Gamma r_4\}$. This proves that η is 4-to-1. $\square$

The vertical and horizontal reflections v, h of the model square σ act on $\mathfrak{R}$ and generate a group $\Sigma \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of symmetries of $\mathfrak{R}$. The Σ-orbit of each $r \in \mathfrak{R}$ contains four elements. Choose once and for all a subset $\mathfrak{R}^+ \subset \mathfrak{R}$ containing precisely one element from each Σ-orbit. The map η restricts to a 1-1 correspondence between $\mathfrak{R}^+$ and the set of geometric squares X^2. For each $\phi \in \Sigma - \{1\}$, let $\mathfrak{R}^\phi$ denote the image of $\mathfrak{R}^+$ under ϕ. Then $\mathfrak{R}$ may be expressed as a disjoint union

$$\mathfrak{R} = \mathfrak{R}^+ \cup \mathfrak{R}^v \cup \mathfrak{R}^h \cup \mathfrak{R}^{vh}.$$

Now we formalize the notion of horizontal and vertical directed edges in X. Consider the two directed edges $[\mathbf{00}, \mathbf{10}], [\mathbf{00}, \mathbf{01}]$ of the model square σ.

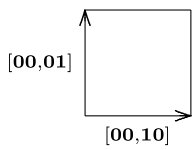


FIGURE 3. Directed edges of the model square σ.

Let A be the set of type rotating isometries $r : [\mathbf{00}, \mathbf{10}] \to \Delta$, and let B be the set of type rotating isometries $r : [\mathbf{00}, \mathbf{01}] \to \Delta$. There is a natural 2-to-1 mapping $r \mapsto \operatorname{range} r$, from $A \cup B$ onto Δ^1. Let $\mathfrak{A} = \Gamma\backslash A$ and $\mathfrak{B} = \Gamma\backslash B$. Call $\mathfrak{A}, \mathfrak{B}$ the sets of horizontal and vertical *directed edges* of $X = \Gamma\backslash\Delta$. Let $\mathcal{E} = \mathfrak{A} \cup \mathfrak{B}$, the set of all directed edges of X.

If $a = \Gamma r \in \mathfrak{A}$, let $o(a) = \Gamma r(\mathbf{00}) \in X^0$ and $t(a) = \Gamma r(\mathbf{10}) \in X^0$, the *origin* and *terminus* of the directed edge a. Similarly, if $b = \Gamma r \in \mathfrak{B}$, let $o(b) = \Gamma r(\mathbf{00}) \in X^0$ and $t(b) = \Gamma r(\mathbf{01}) \in X^0$. Note that it is possible that $o(e) = t(e)$.

A straightforward analogue of Lemma 2.2 shows that each geometric edge in X^1 is the image of each of two directed edges. The horizontal and vertical reflections on σ induce an inversion on $\mathcal{E}$, denoted by $e \mapsto \overline{e}$, with the property that $\overline{\overline{e}} = e$ and $o(e) = t(\overline{e})$. The pair $(\mathcal{E}, X^0)$ is thus a graph in the sense of [Se]. Choose once and for all an orientation of this graph: that is a subset $\mathcal{E}^+$ of $\mathcal{E}$, with $\mathcal{E} = \mathcal{E}^+ \sqcup \overline{\mathcal{E}^+}$. Write $\mathfrak{A}^+ = \mathfrak{A} \cap \mathcal{E}^+$ and $\mathfrak{B}^+ = \mathfrak{B} \cap \mathcal{E}^+$. The images of $\mathfrak{A}$ [respectively $\mathfrak{B}$] in X^1 are the edges the *horizontal [vertical] 1-skeleton* X^1_h $[X^1_v]$.

Lemma 2.3. *There is a well defined injective map*

$$t \mapsto (a(t), b(t)) : \mathfrak{R} \to \mathfrak{A} \times \mathfrak{B}$$

which is surjective if X has one vertex.

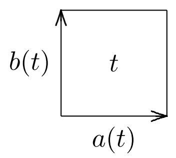


FIGURE 4. Directed edges in X.

Proof. The map $r \mapsto (r|_{[\mathbf{00},\mathbf{10}]}, r|_{[\mathbf{00},\mathbf{01}]}) : R \to A \times B$ is injective because each geometric square of Δ is uniquely determined by any two edges containing a common vertex.

If $t = \Gamma r \in \mathfrak{R}$ then define

$$a(t) = \Gamma r|_{[\mathbf{00},\mathbf{10}]}, \quad b(t) = \Gamma r|_{[\mathbf{00},\mathbf{01}]}.$$

Using the fact that Γ acts freely on Δ it is easy to see that the map $t \mapsto (a(t), b(t))$ is injective.

If X has one vertex, then any two elements $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ are represented by type rotating isometries $r_1 : [\mathbf{00}, \mathbf{10}] \to \Delta$, $r_2 : [\mathbf{00}, \mathbf{01}] \to \Delta$ with $r_1(\mathbf{00}) = r_2(\mathbf{00})$. The isometries r_1, r_2 are restrictions of an isometry $r \in R$, which defines an element $t = \Gamma r \in \mathfrak{R}$ with $a = a(t)$ and $b = b(t)$. $\square$

If $t = \Gamma r \in \mathfrak{R}$, define directed edges $a'(t) \in \mathfrak{A}, b'(t) \in \mathfrak{B}$ opposite to $a(t), b(t)$, as follows.

$$\begin{aligned} a'(t) &= \Gamma(r \circ v|_{[\mathbf{00},\mathbf{10}]}), \\ b'(t) &= \Gamma(r \circ h|_{[\mathbf{00},\mathbf{01}]}). \end{aligned}$$

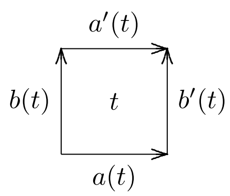


FIGURE 5. Opposite edges.

In other words

$$a'(t) = a(t^v); \quad b'(t) = b(t^h). \tag{4}$$

3. Some related graphs

Associated to the VH-T complex X are two graphs whose vertices are directed edges of X. Denote by $\mathcal{G}_v(\mathfrak{A})$ the graph whose vertex set is $\mathfrak{A}$ and whose edge set is $\mathfrak{R}$, with origin and terminus maps defined by $t \mapsto a(t)$ and $t \mapsto a'(t)$ respectively. Similarly $\mathcal{G}_h(\mathfrak{B})$ is the graph whose vertex set is $\mathfrak{B}$ and whose edge set is $\mathfrak{R}$, with the origin and terminus maps defined by $t \mapsto b(t)$ and $t \mapsto b'(t)$.

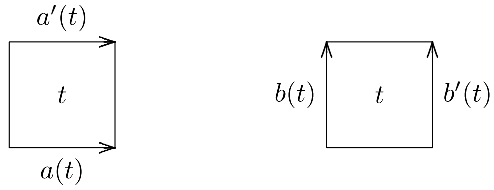


FIGURE 6. Edges of $\mathcal{G}_v(\mathfrak{A})$ and $\mathcal{G}_h(\mathfrak{B})$.

Now define two *directed* graphs whose vertices are elements of $\mathfrak{R}$. The "horizontal" graph $\mathcal{G}_h(\mathfrak{R})$ has vertex set $\mathfrak{R}$. A directed edge $[t, s]$ is defined as follows. Consider the model rectangle H made up of two adjacent squares with vertices $\{(i, j) \in \mathbb{Z}^2 : i = 0, 1, 2, j = 0, 1\}$ where the vertex (i, j) has type $(i + 2\mathbb{Z}, j + 2\mathbb{Z})$. The model square σ of Figure 2 is considered as the left hand square of H.

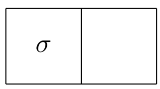
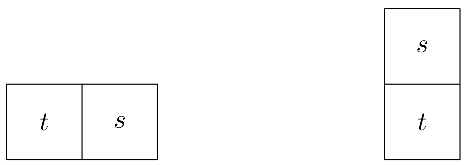


FIGURE 7. The model rectangle H.

An isometry $r : H \to \Delta$ is said to be type rotating if there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex (i, j) of H, $\tau(r((i, j))) = (i + k, j + l)$. A *directed edge* of $\mathcal{G}_h(\mathfrak{R})$ is Γr where $r : H \to \Delta$ be a type rotating isometry. The *origin* of Γr is $t = \Gamma r_1$, where $r_1 = r|_\sigma$ and the *terminus* of Γr is $s = \Gamma r_2$, where $r_2 : \sigma \to \Delta$ is defined by $r_2(i, j) = r(i + 1, j)$. There is a similar definition for the "vertical" graph $\mathcal{G}_v(\mathfrak{R})$ with vertex set $\mathfrak{R}$. Edges $[t, s]$ of $\mathcal{G}_h(\mathfrak{R})$ and $\mathcal{G}_v(\mathfrak{R})$ are illustrated in Figure 8, by the ranges of representative isometries.



An edge of $\mathcal{G}_h(\mathfrak{R})$ An edge of $\mathcal{G}_v(\mathfrak{R})$

FIGURE 8

Since Γ acts freely on Δ, it is easy to see that the existence of a directed edge $[t,s]$ of $\mathcal{G}_h(\mathfrak{R})$ with origin $t \in \mathfrak{R}$ and terminus $s \in \mathfrak{R}$ is equivalent to

$$b(s) = b'(t), \quad s \neq t^h. \tag{5}$$

Similarly the existence of a directed edge $[t,s]$ of $\mathcal{G}_v(\mathfrak{R})$, with origin $t \in \mathfrak{R}$ and terminus $s \in \mathfrak{R}$ is equivalent to

$$a(s) = a'(t), \quad s \neq t^v. \tag{6}$$

The next Lemma will be used later. Recall that a lattice Γ in $\mathrm{PGL}_2(\mathbb{Q}_p)\times\mathrm{PGL}_2(\mathbb{Q}_\ell)$ is automatically cocompact [Mar, IX Proposition 3.7)].

Lemma 3.1. *If p, ℓ are prime and Γ is a torsion free irreducible lattice in $\mathrm{PGL}_2(\mathbb{Q}_p)\times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ acting on the corresponding product of trees, then the directed graphs $\mathcal{G}_h(\mathfrak{R})$, $\mathcal{G}_v(\mathfrak{R})$ are connected.*

Proof. This follows from [M3, Proposition 2.15], using the topological transitivity of an associated shift system. The proof uses the Howe-Moore theorem for p-adic semisimple groups and is explained in [M2, Lemma 2]. $\square$

4. Tilings and $H_2(\Gamma, \mathbb{Z})$

Throughout this section, $\mathcal{T}_1$ and $\mathcal{T}_2$ are locally finite trees whose vertices all have degree at least three. The group Γ acts freely and cocompactly on the 2 dimensional cell complex $\Delta = \mathcal{T}_1 \times \mathcal{T}_2$ and we continue to use the notation introduced in the preceding sections.

For $t, s \in \mathfrak{R}$ write tHs [respectively tVs] to mean that there is a "horizontal" [respectively "vertical"] directed edge $[t,s]$ in $\mathcal{G}_h(\mathfrak{R})$ [respectively $\mathcal{G}_v(\mathfrak{R})$]. Define homomorphisms $T_1, T_2 : \mathbb{Z}\mathfrak{R} \to \mathbb{Z}\mathfrak{R}$ by

$$T_1 t = \sum_{tHs} s, \qquad T_2 t = \sum_{tVs} s.$$

It follows from (5),(6) that

$$T_1 t = \left(\sum_{b(s)=b'(t)} s \right) - t^h,$$

$$T_2 t = \left(\sum_{a(s)=a'(t)} s \right) - t^v.$$

Consider the homomorphism

$$\begin{pmatrix} T_1 - I \\ T_2 - I \end{pmatrix} : \quad \mathbb{Z}\mathfrak{R} \to \mathbb{Z}\mathfrak{R} \oplus \mathbb{Z}\mathfrak{R},$$

$$t \mapsto (T_1 t - t) \oplus (T_2 t - t).$$

Define $\varepsilon : \mathbb{Z}\mathcal{E} \to \mathbb{Z}\mathcal{E}^+$ by

$$\varepsilon(x) = \begin{cases} x & \text{if } x \in \mathcal{E}^+, \\ -\overline{x} & \text{if } x \in \overline{\mathcal{E}^+}. \end{cases}$$

The boundary map $\partial : \mathbb{Z}\mathfrak{R}^+ \to \mathbb{Z}\mathcal{E}^+$ is defined by

$$\partial t = \varepsilon(a(t) + b'(t) - a'(t) - b(t))$$

and since X is 2-dimensional, $H_2(\Gamma,\mathbb{Z})=\ker\partial$. Define a homomorphism

$$\varphi_2:\mathbb{Z}\mathfrak{R}^+\to\mathbb{Z}\mathfrak{R}$$

by

$$\varphi_2 t = t - t^v - t^h + t^{vh}.$$

The rest of this section is devoted to proving the following result, which is a more precise version of Theorem 1.1.

Theorem 4.1. *The homomorphism φ_2 restricts to an isomorphism from $H_2(\Gamma,\mathbb{Z})$ onto* $\ker\left(\begin{smallmatrix}T_1-I\\T_2-I\end{smallmatrix}\right)$.

Define a homomorphism $\varphi_1:\mathbb{Z}\mathcal{E}\to\mathbb{Z}\mathfrak{R}\oplus\mathbb{Z}\mathfrak{R}$ by

$$\varphi_1(a)=0\oplus\left(\sum_{a(s)=\overline{a}} s-\sum_{a(s)=a} s\right),\qquad \text{if } a\in\mathfrak{A},$$

$$\varphi_1(b)=\left(\sum_{b(s)=b} s-\sum_{b(s)=\overline{b}} s\right)\oplus 0,\qquad \text{if } b\in\mathfrak{B}.$$

Note that if $x\in\mathcal{E}$ then $\varphi_1(\overline{x})=-\varphi_1(x)$ and so $\varphi_1(\varepsilon(x))=\varphi_1(x)$.

Lemma 4.2. *The homomorphisms φ_1,φ_2 are injective and the following diagram commutes:*

$$\begin{array}{ccc}
\mathbb{Z}\mathcal{E}^+ & \xleftarrow{\ \partial\ } & \mathbb{Z}\mathfrak{R}^+\\
{\scriptstyle\varphi_1}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_2}\\
\mathbb{Z}\mathfrak{R}\oplus\mathbb{Z}\mathfrak{R} & \xleftarrow[\left(\begin{smallmatrix}T_1-I\\T_2-I\end{smallmatrix}\right)]{} & \mathbb{Z}\mathfrak{R}
\end{array}\tag{7}$$

Proof. Let $t\in\mathfrak{R}$. Then

$$(T_1-I)t=\left(\sum_{b(s)=b'(t)} s\right)-t^h-t,$$

$$(T_1-I)t^v=\left(\sum_{b(s)=\overline{b'(t)}} s\right)-t^{vh}-t^v,$$

$$(T_1-I)t^h=\left(\sum_{b(s)=b(t)} s\right)-t-t^h,$$

$$(T_1-I)t^{vh}=\left(\sum_{b(s)=\overline{b(t)}} s\right)-t^v-t^{vh}.$$

Therefore

$$\begin{aligned}(T_1-I)\circ\varphi_2(t)&=(T_1-I)(t-t^v-t^h+t^{vh})\\
&=\left(\sum_{b(s)=b'(t)} s-\sum_{b(s)=\overline{b'(t)}} s\right)-\left(\sum_{b(s)=b(t)} s-\sum_{b(s)=\overline{b(t)}} s\right).\end{aligned}$$

By definition of φ_1, this implies that

$$\varphi_1(b'(t) - b(t)) = (T_1 - I)\varphi_2(t) \oplus 0.$$

Similarly

$$\varphi_1(a(t) - a'(t)) = 0 \oplus (T_2 - I)\varphi_2(t).$$

Therefore

$$\begin{aligned} \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right) \circ \varphi_2(t) &= \varphi_1(b'(t) - b(t) + a(t) - a'(t)) \\ &= \varphi_1 \circ \varepsilon(b'(t) - b(t) + a(t) - a'(t)) \\ &= \varphi_1 \circ \partial(t). \end{aligned}$$

This shows that (7) commutes.

It is obvious that φ_2 is injective. To verify that φ_1 is injective, define $\psi : \mathbb{Z}\mathfrak{R} \oplus \mathbb{Z}\mathfrak{R} \to \mathbb{Z}\mathcal{E}^+$ by $\psi(s,t) = \varepsilon(b(s) - a(t))$. Then $\psi \circ \varphi_1(x)$ is a nonzero multiple of x, for all $x \in \mathcal{E}$. It follows that $\psi \circ \varphi_1 : \mathbb{Z}\mathcal{E}^+ \to \mathbb{Z}\mathcal{E}^+$ is injective and therefore so is φ_1. □

Lemma 4.3. *The homomorphism φ_2 restricts to an isomorphism from $H_2(\Gamma, \mathbb{Z})$ onto $\varphi_2(\mathbb{Z}\mathfrak{R}^+) \cap \ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right)$.*

Proof. Let $\varphi_2(\beta) \in \ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right)$, where $\beta \in \mathbb{Z}\mathfrak{R}^+$. It follows from (7) that

$$\varphi_1 \circ \partial(\beta) = 0\,.$$

But φ_1 is injective, so $\partial\beta = 0$ i.e. $\beta \in H_2(\Gamma, \mathbb{Z})$.

Conversely, if $\beta \in H_2(\Gamma, \mathbb{Z})$ then $\left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right) \circ \varphi_2(\beta) = 0$ by (7), so

$$\varphi_2(\beta) \in \ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right).$$

Since φ_2 is injective, the conclusion follows. □

The next result, combined with Lemma 4.3, completes the proof of Theorem 4.1.

Lemma 4.4. *There is an inclusion* $\ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right) \subset \varphi_2(\mathbb{Z}\mathfrak{R}^+)$.

Proof. Let $\alpha = \sum_{t \in \mathfrak{R}} \lambda(t) t \in \ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right)$. We show that $\alpha \in \varphi_2(\mathbb{Z}\mathfrak{R}^+)$. If $s \in \mathfrak{R}$ then the coefficient of s in the sum representing $(T_1 - I)\alpha$ is

$$\left(\sum_{\substack{t \in \mathfrak{R}, t \neq s^h \\ b'(t) = b(s)}} \lambda(t) \right) - \lambda(s) = \left(\sum_{\substack{t \in \mathfrak{R} \\ b'(t) = b(s)}} \lambda(t) \right) - \lambda(s) - \lambda(s^h).$$

This coefficient is zero, since $\alpha \in \ker(T_1 - I)$. Therefore

$$\lambda(s) + \lambda(s^h) = \sum_{\substack{t \in \mathfrak{R} \\ b'(t) = b(s)}} \lambda(t). \tag{8}$$

The right hand side of equation (8) depends only on $b(s)$, so for any $b \in \mathfrak{B}$ we define

$$\mu(b) = \sum_{\substack{t \in \mathfrak{R} \\ b'(t) = b}} \lambda(t).$$

Thus (8) may be rewritten as

$$\lambda(s) + \lambda(s^h) = \mu(b(s)). \tag{9}$$

It follows from (8) and (4) that

$$\mu(b(s)) = \mu(b(s^h)) = \mu(b'(s)). \tag{10}$$

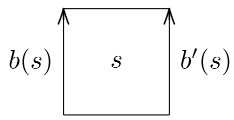


FIGURE 9. $\mu(b(s)) = \mu(b'(s))$

Fix an element $b_0 \in \mathfrak{B}$, and let $\mathcal{C}$ be the connected component of the graph $\mathcal{G}_h(\mathfrak{B})$ containing b_0. Then $\mathcal{C}$ is a connected graph with vertex set $\mathcal{C}^0 \subset \mathfrak{B}$ and edge set $\mathcal{C}^1 \subset \mathfrak{R}$. The graph $\mathcal{C}$ has a natural orientation $\mathcal{C}^+ = \mathcal{C}^1 \cap (\mathfrak{R}^+ \cup \mathfrak{R}^v)$ and it is clear that $\mathcal{C}^1 = \mathcal{C}^+ \cup \{t^h : t \in \mathcal{C}^+\}$. Each vertex of $\mathcal{C}$ has degree at least three, since the same is true of the tree $\mathcal{T}_1$. Therefore the number of vertices of $\mathcal{C}$ is less than the number of geometric edges i.e. $|\mathcal{C}^0| < |\mathcal{C}^+|$.

If $b \in \mathcal{C}^0$ then there is a path in $\mathcal{C}^0$ from b_0 to b. It follows by induction from (10) that $\mu(b_0) = \mu(b)$. Thus

$$\mu(b_0) = \sum_{\substack{t \in \mathfrak{R} \\ b'(t) = b}} \lambda(t) = \sum_{\substack{t \in \mathcal{C}^1 \\ b'(t) = b}} \lambda(t).$$

Therefore

$$\begin{aligned} |\mathcal{C}^0| \mu(b_0) &= \sum_{b \in \mathcal{C}^0} \sum_{\substack{t \in \mathcal{C}^1 \\ b'(t) = b}} \lambda(t) = \sum_{t \in \mathcal{C}^1} \lambda(t) \\ &= \sum_{t \in \mathcal{C}^+} (\lambda(t) + \lambda(t^h)) = \sum_{t \in \mathcal{C}^+} \mu(b(t)) \\ &= \sum_{t \in \mathcal{C}^+} \mu(b_0) = |\mathcal{C}^+| \mu(b_0). \end{aligned}$$

Since $|\mathcal{C}^0| < |\mathcal{C}^+|$, it follows that $\mu(b_0) = 0$ for all $b_0 \in \mathfrak{B}$. In other words, by (9),

$$\lambda(s) = -\lambda(s^h) \tag{11}$$

for all $s \in \mathfrak{R}$. A similar argument, using $\alpha \in \ker(T_2 - I)$ and interchanging the roles of horizontal and vertical reflections, shows that

$$\lambda(s) = -\lambda(s^v) \tag{12}$$

for all $s \in \mathfrak{R}$. Combining (11) and (12) gives

$$\lambda(s) = \lambda(s^{vh}) \tag{13}$$

for all $s \in \mathfrak{R}$. Finally,

$$\begin{aligned}\alpha &= \sum_{t\in\mathfrak{R}^+} \left(\lambda(s)s + \lambda(s^v)s^v + \lambda(s^h)s^h + \lambda(s^{vh})s^{vh}\right)\\ &= \sum_{t\in\mathfrak{R}^+} \lambda(s)\left(s - s^v - s^h + s^{vh}\right)\\ &= \sum_{t\in\mathfrak{R}^+} \lambda(s)\varphi_2(s) \in \varphi_2(\mathbb{Z}\mathfrak{R}^+).\end{aligned}$$

□

5. K-theory of the boundary C^*-algebra

The (maximal) boundary $\partial\Delta$ of Δ is defined in [KR]. It is homeomorphic to $\partial\mathcal{T}_1 \times \partial\mathcal{T}_2$, where $\partial\mathcal{T}_j$ is the totally disconnected space of ends of the tree $\mathcal{T}_j$. The group Γ acts on $\partial\Delta$ and hence on $C_{\mathbb{C}}(\partial\Delta)$ via $g \mapsto \alpha_g$, where $\alpha_g f(\omega) = f(g^{-1}\omega)$, for $f \in C_{\mathbb{C}}(\partial\Delta)$, $g \in \Gamma$. The full crossed product C^*-algebra $\mathcal{A}(\Gamma, \partial\Delta) = C_{\mathbb{C}}(\partial\Delta) \rtimes \Gamma$ is the completion of the algebraic crossed product in an appropriate norm. We present examples where the rank of the analytic K-group $K_0(\mathcal{A}(\Gamma, \partial\Delta))$ is determined by Theorem 4.1.

5.1. **One vertex complexes.** The case where the quotient VH-T complex X has one vertex was studied in [KR]. The group Γ acts freely and transitively on the vertices of Δ and $\mathcal{A}(\Gamma, \partial\Delta)$ is isomorphic to a rank-2 Cuntz-Krieger algebra, as described in [RS1, RS2]. The proof of this fact given in [KR, Theorem 5.1]. It follows from [RS1] that $\mathcal{A}(\Gamma, \partial\Delta)$ is classified by its K-theory. By the proofs of [RS2, Proposition 4.13] and [KR, Lemma 4.3, Theorem 5.3], we have

$$K_0(\mathcal{A}(\Gamma, \partial\Delta)) = K_1(\mathcal{A}(\Gamma, \partial\Delta))$$

and

$$\operatorname{rank}(K_0(\mathcal{A}(\Gamma, \partial\Delta))) = 2 \cdot \dim\ker \left(\begin{smallmatrix} T_1 - I \\ T_2 - I \end{smallmatrix}\right).$$

Together with Theorem 4.1, this proves

$$\operatorname{rank} K_0(\mathcal{A}(\Gamma, \partial\Delta)) = 2 \cdot \operatorname{rank} H_2(\Gamma, \mathbb{Z}). \tag{14}$$

This verifies a conjecture in [KR].

5.2. **Irreducible lattices in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$.** If p, ℓ are prime then the group $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$ acts on the $\Delta = \mathcal{T}_{p+1} \times \mathcal{T}_{\ell+1}$ and on its boundary $\partial\Delta$, which can be identified with a direct product of projective lines $\mathbb{P}_1(\mathbb{Q}_p) \times \mathbb{P}_1(\mathbb{Q}_\ell)$. Let Γ be a torsion free irreducible lattice in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_\ell)$. Then Γ acts freely on Δ and $\mathcal{A}(\Gamma, \partial\Delta)$ is a rank-2 Cuntz-Krieger algebra in the sense of [RS1]. The irreducibility condition (H2) of [RS1] follows from Lemma 3.1. The proofs of the remaining conditions of [RS1] are exactly the same as in [KR, Lemma 4.1]. It follows that (14) is also true in this case. Since Γ is irreducible, the normal subgroup theorem [Mar, IV, Theorem (4.9)] implies that $H_1(\Gamma, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ is finite. Equation (14) can therefore be written

$$\chi(\Gamma) = 1 + \frac{1}{2}\operatorname{rank} K_0(\mathcal{A}(\Gamma, \partial\Delta)). \tag{15}$$

On the other hand, one easily calculates

$$\chi(\Gamma) = \frac{(p-1)(\ell-1)}{4}|X^0|$$

where $|X^0|$ is the number of vertices of X. Therefore the rank of $K_0(\mathcal{A}(\Gamma, \partial\Delta))$ can be expressed explicitly in terms of p, ℓ and $|X^0|$.

Explicit examples are studied in [M3, Section 3]. If $p, l \equiv 1 \pmod 4$ are two distinct primes, Mozes constructs an irreducible lattice $\Gamma_{p,\ell}$ in $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$ which acts freely and transitively on the vertex set of Δ. Here is how $\Gamma_{p,l}$ is constructed. Let $\mathbb{H}(\mathbb{Z}) = \{a = a_0 + a_1 i + a_2 j + a_3 k; a_j \in \mathbb{Z}\}$, the ring of integer quaternions, let i_p be a square root of -1 in $\mathbb{Q}_p$ and define

$$\psi : \mathbb{H}(\mathbb{Z}) \to PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_\ell)$$

by

$$\psi(a) = \left(\begin{bmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{bmatrix}, \begin{bmatrix} a_0 + a_1 i_\ell & a_2 + a_3 i_\ell \\ -a_2 + a_3 i_\ell & a_0 - a_1 i_\ell \end{bmatrix} \right).$$

Let $\tilde{\Gamma}_{p,\ell} = \{a \in \mathbb{H}(\mathbb{Z}); a_0 \equiv 1 \pmod 2, a_j \equiv 0 \pmod 2, j = 1,2,3, |a|^2 = p^r l^s\}$. Then $\Gamma_{p,\ell} = \psi(\tilde{\Gamma}_{p,\ell})$. The fact that $\Gamma_{p,\ell}$ is irreducible follows easily from [RR, Corollary 2.3], where it is observed that the only nontrivial direct product subgroup of $\Gamma_{p,\ell}$ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Since $|X^0| = 1$, it follows from (15) that

$$\operatorname{rank} K_0(\mathcal{A}(\Gamma, \partial\Delta)) = \frac{(p-1)(\ell-1)}{2} - 2.$$

This proves an experimental observation of [KR, Example 6.2]. The construction of Mozes has been generalized in [Rat, Chapter 3] to all pairs (p, l) of distinct odd primes and the same conclusion applies.

References

[BM] M. Burger and S. Mozes, Lattices in products of trees, *Inst. Hautes Études Sci. Publ. Math.* **92** (2001), 151–194.

[KR] J. Kimberley and G. Robertson, Groups acting on products of trees, tiling systems and analytic K-theory, *New York J. Math.* **8** (2002), 111–131.

[M2] S. Mozes, On closures of orbits and arithmetic of quaternions, *Israel J. Math.* **86** (1994), 195–210.

[M3] S. Mozes, Actions of Cartan subgroups, *Israel J. Math.* **90** (1995), 253–294.

[Mar] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, Berlin, 1991.

[Rat] D. Rattaggi, *Computations in groups acting on a product of trees: normal subgroup structures and quaternion lattices*, Ph.D. thesis, ETH Zürich, 2004.

[RR] D. Rattaggi and G. Robertson, Abelian subgroup structure of square complex groups and arithmetic of quaternions, *J. Algebra* **286** (2005), 57–68.

[RS1] G. Robertson and T. Steger, Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras, *J. reine angew. Math.* **513** (1999), 115–144.

[RS2] G. Robertson and T. Steger, Asymptotic K-theory for groups acting on $\widetilde{A}_2$ buildings, *Canad. J. Math.* **53** (2001), 809–833.

[Se] J.-P. Serre, *Arbres, amalgames, SL_2*, Astérisque **46**, Soc. Math. France, 1977.

[Ti] J. Tits, *Reductive groups over local fields*, Proceedings of Symposia in Pure Mathematics **33** (1979), 29–69.

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